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PLURIPOLAR HULLS AND HOLOMORPHIC COVERINGS

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ABSTRACT

The aim of the paper is to study the behavior of (complete) pluripolar sets under special holomorphic mappings (proper mappings and coverings).

1. Introduction

Let D be a domain in $\mathbb{C}^n$. We denote by $\mathrm{PSH}(D)$ the set of all plurisubharmonic functions in D. We assume that the constant function $-\infty$ is a plurisubharmonic function. Following P. Lelong (see [Lel]), we say that a set $P \subset D$ is **pluripolar** if for any point $z_0 \in D$ there is a connected open neighborhood U of z_0 and a plurisubharmonic function $u \in \mathrm{PSH}(U)$, $u \not\equiv -\infty$, such that $P \cap U \subset \{z \in U : u(z) = -\infty\}$. According to Josefson (see [Jos]), the local definition of pluripolarity coincides with the global definition, i.e., for any pluripolar set $P \subset D$ there exists a plurisubharmonic function $u \in \mathrm{PSH}(D)$, $u \not\equiv -\infty$, such that

$$P \subset \{z \in D : u(z) = -\infty\}. \tag{1}$$

In the case of equality in (1), we say that P is **complete pluripolar**. For $n = 1$, we call P polar and complete polar, respectively.

It is easy to see that any complete pluripolar set must be of G_δ-type. By Deny's theorem (see, e.g., [Lan]) for any polar set P in $\mathbb{C}$ of G_δ-type there exists a

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subharmonic function $u \in \mathrm{SH}(\mathbb{C})$ such that $P = \{x \in \mathbb{C} : u(x) = -\infty\}$. In higher dimension the situation is more complicated. Take $P = V \times \{0\} \subset \mathbb{C}^2$, where V is a non-polar subset of $\mathbb{C}$. Then any plurisubharmonic function $u \in \mathrm{PSH}(\mathbb{C}^2)$ which is equal to $-\infty$ on P is to equal $-\infty$ on the set $\mathbb{C} \times \{0\}$. To describe this phenomenon of "propagation" of pluripolar sets N. Levenberg and E. A. Poletsky [Lev-Pol] considered two types of pluripolar hulls of a pluripolar set P in a domain D:

$$P_D^* := \bigcap \{z \in D : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions in D which are $-\infty$ on P, and

$$P_D^- := \bigcap \{z \in D : u(z) = -\infty\},$$

where the intersection is taken over all *negative* plurisubharmonic functions in D which are $-\infty$ on P. Note that $P_D^* \subset P_D^-$.

There is a nice relation between these two pluripolar hulls.

THEOREM 1 (see [Lev-Pol]): *Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let D_ν be an increasing sequence of relatively compact domains with $\bigcup_{\nu=1}^{\infty} D_\nu = D$. Let $P \subset D$ be pluripolar. Then*

$$P_D^* = \bigcup_{\nu=1}^{\infty} (P \cap D_\nu)_{D_\nu}^- .$$

Moreover, if D is a hyperconvex domain, then $P_D^* = P_D^-$. Recall that a bounded domain $D \subset \mathbb{C}^n$ is called hyperconvex if there exists a negative plurisubharmonic exhaustion function u of D, i.e., $\{z \in D : u(z) < \beta\}$ is relatively compact in D for any $\beta < 0$ (see, e.g., [Kli]).

Note that for any complete pluripolar set P in D we have $P_D^* = P$. As a partial converse of this remark we have the following result.

THEOREM 2 (see [Zer]): *Let D be a pseudoconvex domain in $\mathbb{C}^n$. Assume that $P \subset D$ is of F_σ-type. Then P is a complete pluripolar set in D if and only if P is of G_δ-type and $P_D^* = P$.*

Theorem 2 shows the importance of pluripolar hulls in the study of complete pluripolarity. Another advantage is the relation between pluripolar hulls and the relative extremal function. For a domain D in $\mathbb{C}^n$ and any subset $E \subset D$ we define the **relative extremal function** as follows:

$$\omega(z, E, D) := \sup\{u(x) : u \in \mathrm{PSH}(D),\ u \le -1 \text{ on } E,\ u < 0 \text{ on } D\}.$$

We have the following very useful result.

THEOREM 3 (see [Lev-Pol]): *Let D be a domain in $\mathbb{C}^n$ and let $P \subset D$ be a pluripolar set. Then*

$$P_D^- = \{z \in D : \omega(z, P, D) < 0\}.$$

Using the above-mentioned results we study the behavior of pluripolar hulls under special holomorphic mappings (proper mappings and coverings).

THEOREM 4: *Let D, G be domains in $\mathbb{C}^n$ and let $h : D \to G$ be a proper holomorphic mapping. Then for any pluripolar set $P \subset D$ we have*

$$h(P)_G^* = h(P_D^*) \quad \text{and} \quad h(P)_G^- = h(P_D^-).$$

Moreover, if P is a complete pluripolar set in D, then $h(P)$ is a complete pluripolar set in G.

The proof of Theorem 4 is given in Section 2.

As we see, the invariance of pluripolar hulls under proper holomorphic mappings is quite clear. The situation with holomorphic coverings seems to be much more complicated. We introduce a class of holomorphic coverings (A-coverings) for which we are able to say more on pluripolar hulls. As a corollary we solve the problem of invariance for product type domains. Namely, we have the following.

THEOREM 5: *Let $D_1, \ldots, D_n$, $G_1, \ldots, G_n$ be domains in $\mathbb{C}$ such that $\partial G_1, \ldots, \partial G_n$ are non-polar. Assume that h_j: $D_j \to G_j$, $j = 1, \ldots, n$, are holomorphic coverings. Put $D := D_1 \times \cdots \times D_n$, $G := G_1 \times \cdots \times G_n$, and $h := (h_1, \ldots, h_n)$. Then for any pluripolar set $P \subset D$ we have*

$$h(P)_G^* = h(P)_G^- = h(P_D^-) = h(P_D^*). \tag{2}$$

The proof of Theorem 5 is given in Section 4. Special cases of Theorem 5 could be found in [Lev-Pol], [Wiel]. As we shall see (Example 24) equality (2) in general (i.e., for any holomorphic covering and any pluripolar set) does not hold. We give also some examples and applications of our results.

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2. Proof of Theorem 4

Note that, generally, there is no problem with the preimage of a complete pluripolar set under holomorphic mapping. Using the fact that the composition of a plurisubharmonic function, and a holomorphic mapping is a plurisubharmonic function, we have the following result.

THEOREM 6: *Let D, G be domains in $\mathbb{C}^n$ and let $h : D \to G$ be a non-constant holomorphic mapping. Then:*

(a) *for any pluripolar set $P \subset D$ we have*

$$h(P_D^*) \subset h(P)_G^* \quad \text{and} \quad h(P_D^-) \subset h(P)_G^-,$$

(b) *for any pluripolar set $Q \subset G$ we have*

$$(h^{-1}(Q))_D^* \subset h^{-1}(Q_G^*) \quad \text{and} \quad (h^{-1}(Q))_D^- \subset h^{-1}(Q_G^-).$$

Moreover, if Q is a complete pluripolar set in G, then $h^{-1}(Q)$ is also complete pluripolar in D.

Before we present the proof, recall the following definition and result. Let D, G be domains in $\mathbb{C}^n$ and let $h: D \to G$ be a holomorphic mapping. Suppose that $z_0 \in D$ is an isolated point of the set $h^{-1}(h(z_0))$. It is well-known (see, e.g., [Sha]) that there exist domains $U \subset D$ and $V \subset G$ such that $h^{-1}(h(z_0)) \cap U = \{z_0\}$ and $h|_U: U \to V$ is a proper holomorphic mapping. We denote $m_{z_0}(h)$ the multiplicity of a proper holomorphic mapping $h|_U$.

THEOREM 7 (see [Lár-Sig]): *Let D, G be domains in $\mathbb{C}^n$ and let $h: D \to G$ be a proper holomorphic mapping. Let u be a plurisubharmonic function on D. Then the function defined by the formula*

$$\tilde{u}(w) := \sum_{z \in h^{-1}(w)} m_z(h) u(z)$$

is plurisubharmonic on G.

Proof of Theorem 4: We know that $h(P)_G^* \supset h(P_D^*)$ and $h(P)_G^- \supset h(P_D^-)$. So, for the equality of these sets it is enough to show that $h(P)_G^* \subset h(P_D^*)$ and $h(P)_G^- \subset h(P_D^-)$.

Fix a point $w_0 \notin h(P_D^*)$. Then $h^{-1}(w_0) \cap P_D^* = \emptyset$. Assume that $h^{-1}(w_0) = \{z_1, \dots, z_k\}$. For any $j \in \{1, \dots, k\}$ there is a $u_j \in \mathrm{PSH}(D)$ such that $u_j = -\infty$ on P, $u_j(z_j) > -\infty$.

Put $u := \max\{u_1, \dots, u_k\}$. Note that $u \in \mathrm{PSH}(D)$, $u = -\infty$ on P, and $u(z_j) > -\infty$, $j = 1, \dots, k$. Take $\tilde{u} \in \mathrm{PSH}(G)$ defined in Theorem 7. It is easy to see that $\tilde{u} = -\infty$ on $h(P)$ and $\tilde{u}(w_0) > -\infty$. So, $w_0 \notin h(P)^*_G$.

In a similar way we show that $h(P)^-_G \subset h(P^-_D)$.

The proof is completed by noting that if $P = \{z \in D : u(z) = -\infty\}$, where u is a plurisubharmonic function on D, $u \not\equiv -\infty$, then $\tilde{u}$ defined in Theorem 7 is such that $h(P) = \{w \in G : \tilde{u}(w) = -\infty\}$. ∎

3. Sequences of holomorphic mappings and pluripolar hulls

The main result of this part is the following.

THEOREM 8: *Let D, G be domains in $\mathbb{C}^n$ and let $h\colon D \to G$ be a holomorphic covering. Assume that there exist $\{F_j\}_{j=1}^\infty \subset \mathrm{Aut}(D)$ such that for any $z \in D$ we have $h^{-1}(h(z)) = \bigcup_{j=1}^\infty \{F_j(z)\}$. Then for any pluripolar set $P \subset D$ we have*

$$(h^{-1}(h(P)))^*_D = h^{-1}(h(P^*_D)) \quad \textit{and} \quad (h^{-1}(h(P)))^-_D = h^{-1}(h(P^-_D)).$$

We prove this result in a sequence of lemmas. We start with the investigation of the sequences of pluripolar sets and their pluripolar hulls.

LEMMA 9: *Let D be a domain in $\mathbb{C}^n$ and let $P_j \subset D$ be a sequence of pluripolar sets. Then*

$$(\bigcup_{j=1}^\infty P_j)^*_D = \bigcup_{j=1}^\infty (P_j)^*_D \quad \text{and} \quad (\bigcup_{j=1}^\infty P_j)^-_D = \bigcup_{j=1}^\infty (P_j)^-_D.$$

Proof of Lemma 9: Note that

$$(\bigcup_{j=1}^\infty P_j)^*_D \supset \bigcup_{j=1}^\infty (P_j)^*_D \quad \text{and} \quad (\bigcup_{j=1}^\infty P_j)^-_D \supset \bigcup_{j=1}^\infty (P_j)^-_D.$$

Hence, we shall have established the lemma if we prove the following $(\bigcup_{j=1}^\infty P_j)^*_D \subset \bigcup_{j=1}^\infty (P_j)^*_D$ and $(\bigcup_{j=1}^\infty P_j)^-_D \subset \bigcup_{j=1}^\infty (P_j)^-_D$.

Let $z_0 \notin \bigcup_{j=1}^\infty (P_j)^*_D$. Let $D_j \Subset D_{j+1} \subset D$, $\bigcup_{j=1}^\infty D_j = D$, be an exhaustion of D. For any $j \in \mathbb{N}$ there exists $u_j \in \mathrm{PSH}(D)$ such that $u_j = -\infty$ on P_j and $u_j(z_0) > -\infty$. Take constants $\alpha_j > 0, \beta_j$ such that for a plurisubharmonic function $\widetilde{u}_j := \alpha_j(u_j + \beta_j)$ we have

(i) $\widetilde{u}_j < 0$ on D_j,

(ii) $\widetilde{u}_j(z_0) > -1/2^j$,

(iii) $\widetilde{u}_j = -\infty$ on P_j.

Set $u := \sum_{j=1}^{\infty} \tilde{u}_j$. Then $u \in \mathrm{PSH}(D)$, $u = -\infty$ on $\bigcup_{j=1}^{\infty} P_j$, and $u(z_0) > -1$. Hence, $z_0 \notin (\bigcup_{j=1}^{\infty} P_j)^*_D$.

In a similar way (put $\beta_j = 0$) we show $(\bigcup_{j=1}^{\infty} P_j)^-_D \subset \bigcup_{j=1}^{\infty} (P_j)^-_D$. ∎

Lemma 10: *Let D be a domain in $\mathbb{C}^n$ and let $\{F_j\}_{j=1}^{\infty} \subset \mathrm{Aut}(D)$. Then for any pluripolar set $P \subset D$ we have*

$$(\bigcup_{j=1}^{\infty} F_j(P))^*_D = \bigcup_{j=1}^{\infty} F_j(P^*_D) \quad \text{and} \quad (\bigcup_{j=1}^{\infty} F_j(P))^-_D = \bigcup_{j=1}^{\infty} F_j(P^-_D).$$

Proof of Lemma 10: It follows from Theorem 4 and Lemma 9. ∎

Proof of Theorem 8: It suffices to remark that $h^{-1}(h(P)) = \bigcup_{j=1}^{\infty} F_j(P)$. ∎

The next two results allow us to construct automorphisms like in Theorem 8.

Proposition 11: *Let D, G be domains in $\mathbb{C}^n$ and let D be simply connected. Then for any holomorphic covering $h : D \to G$ there exist $\{F_j\}_{j=1}^{\infty} \subset \mathrm{Aut}(D)$ such that for any $z \in D$ we have $h^{-1}(h(z)) = \bigcup_{j=1}^{\infty} \{F_j(z)\}$.*

Proof of Proposition 11: **Step 1.** Let $z_0 \in D$ be fixed. Then $h^{-1}(h(z_0)) = \{z_0, z_1, z_2, \ldots\}$. For any $j = 0, 1, 2, \ldots$ there exists a holomorphic mapping $F_j \colon D \to D$ such that $h \circ F_j = h$ and $F_j(z_0) = z_j$.

In the same way, there exists a holomorphic mapping $G_j \colon D \to D$ such that $h \circ G_j = h$ and $G_j(z_j) = z_0$. Note that $h \circ F_j \circ G_j(z_j) = h(z_j)$. So, $F_j \circ G_j = \mathrm{id}_D$. Hence, $F_j \in \mathrm{Aut}(D)$, $j = 0, 1, 2, \ldots$.

Step 2. Take any $z \in D$. We have $h \circ F_j(z) = h(z)$, $j = 0, 1, 2, \ldots$. Hence $\bigcup_{j=1}^{\infty} \{F_j(z)\} \subset h^{-1}(h(z))$.

Step 3. Fix $\tilde{z}_0 \neq z_0$. According to Step 1 there are $\tilde{F}_j \in \mathrm{Aut}(D)$ such that $h^{-1}(h(\tilde{z}_0) = \bigcup_{j=1}^{\infty} \{\tilde{F}_j(\tilde{z}_0)\}$.

Fix $j_0 \in \mathbb{N}$. Then according to Step 2, $F_{j_0}(\tilde{z}_0) \in \bigcup_{j=1}^{\infty} \{\tilde{F}_j(\tilde{z}_0)\}$. Hence, there exists $j_1 \in \mathbb{N}$ such that $F_{j_0}(\tilde{z}_0) = \tilde{F}_{j_1}(\tilde{z}_0)$.

Note that $\tilde{F}_{j_1}^{-1} \circ F_{j_0}(\tilde{z}_0) = \tilde{z}_0$ and, therefore, $h \circ \tilde{F}_{j_1}^{-1} \circ F_{j_0}(\tilde{z}_0) = h(\tilde{z}_0)$. Hence, $\tilde{F}_{j_1}^{-1} \circ F_{j_0} = \mathrm{id}_D$ and $\tilde{F}_{j_1} = F_{j_0}$. So, $\{F_j\}_{j=1}^{\infty} \subset \{\tilde{F}_j\}_{j=1}^{\infty}$ and, therefore, $\{F_j\}_{j=1}^{\infty} = \{\tilde{F}_j\}_{j=1}^{\infty}$. ∎

Proposition 12: *Let D_1, D_2, G_1, G_2 be domains and let $h_i \colon D_i \to G_i$, $i = 1, 2$, be holomorphic coverings. Suppose that there exist $\{F_j^i\}_{j=1}^{\infty} \subset \mathrm{Aut}(D_i)$ such that for any $z_i \in D_i$ we have $h_i^{-1}(h_i(z_i)) = \bigcup_{j=1}^{\infty} \{F_j^i(z_i)\}$. Then there exists*

$\{F_j\}_{j=1}^{\infty} \subset \mathrm{Aut}(D_1 \times D_2)$ *such that for any* $z = (z_1, z_2) \in D_1 \times D_2$ *we have* $h^{-1}(h(z)) = \bigcup_{j=1}^{\infty}\{F_j(z)\}$, *where* $h(z) := (h_1(z_1), h_2(z_2))$.

Proof of Proposition 12: It is sufficient to consider $\{(F_j^1, F_k^2)\}_{j,k=1}^{\infty}$. ∎

4. A-coverings

Let D, G be domains in $\mathbb{C}^n$ and let $h\colon D \to G$ be a holomorphic covering. We say that h is an **A-covering** if for any $w_0 \in G$ there exists a neighborhood V_0 of w_0 such that $h^{-1}(V_0) = \bigcup_{j=1}^{\infty} V_j$, where V_j are disjoint open sets, and

$$\lim_{k\to\infty} \omega(z, \bigcup_{j=k}^{\infty} V_j, D) = 0, \quad z \in D.$$

Note that any finite covering is an A-covering. The importance of A-coverings comes from the following result.

THEOREM 13: *Let D, G be domains in $\mathbb{C}^n$ and let $h\colon D \to G$ be an A-covering. Suppose that there exists $\{F_j\}_{j=1}^{\infty} \subset \mathrm{Aut}(D)$ such that for any $z \in D$ we have $h^{-1}(h(z)) = \bigcup_{j=1}^{\infty}\{F_j(z)\}$. Then for any pluripolar set $P \subset D$ we have*

$$h(P)_G^- = h(P_D^-).$$

Moreover, if $P_D^* = P_D^-$ (e.g., D is hyperconvex), then

$$h(P)_G^* = h(P_D^*) = h(P_D^-) = h(P)_G^-.$$

Before we proceed to the proof, recall the following result.

THEOREM 14 (see [Lev-Pol]): *Let D, G be domains in $\mathbb{C}^n$ and let $h\colon D \to G$ be an A-covering. Then for any subset $Q \subset G$ and any $w_0 \in Q$ there exists a neighborhood $V_0 \subset Y$ of w_0 such that*

$$\omega(z, h^{-1}(Q \cap V_0), D) = \omega(h(z), Q \cap V_0, G).$$

Therefore, $[h^{-1}(Q \bigcap V_0)]_D^- = h^{-1}([Q \bigcap V_0]_G^-)$.

For the proof of Theorem 13 we need also the following localization result.

LEMMA 15: *Let D be a domain in $\mathbb{C}^n$ and let $P \subset D$ be a pluripolar set. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be an open covering of P. Then*

$$P_D^* = \bigcup_{\lambda\in\Lambda} (P \cap V_\lambda)_D^* \quad \text{and} \quad P_D^- = \bigcup_{\lambda\in\Lambda} (P \cap V_\lambda)_D^-.$$

Proof of Lemma 15: It follows immediately that $P^*_D \supset \bigcup_{\lambda\in\Lambda}(P\cap V_\lambda)^*_D$ and $P^-_D \supset \bigcup_{\lambda\in\Lambda}(P\cap V_\lambda)^-_D$.

Take a countable set $\{\lambda_j\}_{j=1}^\infty \subset \Lambda$ such that $\bigcup_{j=1}^\infty V_{\lambda_j} \supset P$. According to Lemma 9 we have

$$\bigcup_{\lambda\in\Lambda}(P\cap V_\lambda)^-_D \supset (\bigcup_{j=1}^\infty P\cap V_{\lambda_j})^-_D = P^-_D$$

and

$$\bigcup_{\lambda\in\Lambda}(P\cap V_{\lambda_j})^*_D \supset (\bigcup_{j=1}^\infty P\cap V_{\lambda_j})^*_D = P^*_D. \qquad \blacksquare$$

COROLLARY 16: *Let D, G be domains in $\mathbb{C}^n$ and let $h\colon D\to G$ be an A-covering. Then for any pluripolar set $Q\subset G$ we have*

$$(h^{-1}(Q))^-_D = h^{-1}(Q^-_G).$$

Proof of Corollary 16: The proof is immediate from Theorem 14 and Lemma 15. $\blacksquare$

Proof of Theorem 13: We have

$$h^{-1}(h(P)^-_G) \overset{\text{Cor. 16}}{=} [h^{-1}(h(P))]^-_D \overset{\text{Thm. 8}}{=} h^{-1}(h(P^-_D)).$$

Since h is a covering, $h(P)^-_G = h(P^-_D)$.

Assume that $P^*_D = P^-_D$. Then by Theorem 6 we get

$$h(P)^*_G \supset h(P^*_D) = h(P^-_D) = h(P)^-_G.$$

We know that $h(P)^-_G \supset h(P)^*_G$. Hence, $h(P)^*_G = h(P)^-_G$ and we get the proof. $\blacksquare$

The next result allows us to construct new examples of A-coverings.

PROPOSITION 17: *Let $h_i\colon D_i\to G_i$, $i=1,2$, be holomorphic A-coverings. Then $h=(h_1,h_2)\colon D_1\times D_2\to G_1\times G_2$ is also an A-covering.*

The proof of Proposition 17 follows immediately from the following contractivity of the relative extremal function (see, e.g., [Kli]).

THEOREM 18: *Let $D\subset\mathbb{C}^n$ and $G\subset\mathbb{C}^m$ be domains. Suppose that $h\colon D\to G$ is a non-constant holomorphic function. Then for any subset $E\subset D$ we have*

$$\omega(z,E,D)\geq\omega(h(z),h(E),G), \quad z\in D.$$

Now we give a characterization of A-coverings in $\mathbb{C}$.

THEOREM 19: *Let D, G be domains in $\mathbb{C}$ and let $h\colon D \to G$ be a holomorphic covering. Assume that ∂G is non-polar. Then ∂D is also non-polar and h is an A-covering.*

Proof of Theorem 19: Recall that for a domain $\Omega \subset \mathbb{C}$ we have $\partial\Omega$ is non-polar if and only if for any point $p \in \Omega$, $g_\Omega(\cdot, p) > -\infty$ on $\Omega \setminus \{p\}$ (and this is equivalent with $g_\Omega \not\equiv -\infty$) (see, e.g., [Ran]).

Hence, for any $p \in G$ we have $g_G(\cdot, p) > -\infty$ on $G \setminus \{p\}$. But it is well-known that $g_D(z,q) \geq g_G(f(z), f(q)) \not\equiv -\infty$ for any $q \in D$ (see, e.g., [Ran]). So, ∂D is non-polar.

Fix w_0 and take $R > 0$ such that $h^{-1}(\mathbb{D}(w_0, R)) = \bigcup_{j=1}^{\infty} U_j$, where $\mathbb{D}(w_0, R) = \{w \in \mathbb{C} : |w - w_0| < R\}$ and U_j are disjoint domains. Take any $r \in (0, R)$. Now we proceed as in [Edi] to show that

$$\omega(z, \bigcup_{j=k}^{\infty} V_j, D) \geq \frac{g_D(z, \{w_j\}_{j=k}^{\infty})}{\log(R/r)}, \quad z \in D,$$

where $\bigcup_{j=1}^{\infty} V_j = h^{-1}(\mathbb{D}(w_0, r))$, $h^{-1}(w_0) = \{w_j\}_{j=1}^{\infty}$, and $g_D(\cdot, \{w_j\}_{j=1}^{\infty})$ denotes the Green function with poles at $\{w_j\}_{j=1}^{\infty}$. Consider the subharmonic function

$$u(z) := \frac{g_D(z, \{w_j\}_{j=k}^{\infty})}{\log(R/r)}, \quad z \in D.$$

Note that $u \leq 0$ on D. Take $z \in V_j$, $j \geq k$. Then

$$u(z) \leq \frac{g_{U_j}(z, w_j)}{\log(R/r)} = \frac{g_{\mathbb{D}(w_0,R)}(h(z), w_0)}{\log(R/r)} = \frac{\log \frac{|h(z) - w_0|}{R}}{\log(R/r)} \leq -1.$$

Hence

$$\omega(z, \bigcup_{j=k}^{\infty} V_j, D) \geq u(z) = \frac{g_D(z, \{w_j\}_{j=k}^{\infty})}{\log(R/r)}, \quad z \in D.$$

It is well-known that

$$\sum_{j=1}^{\infty} g_D(z, w_j) = g_D(z, \{w_j\}_{j=1}^{\infty}) \geq g_G(h(z), w_0) > -\infty, \quad z \notin \{w_j\}_{j=1}^{\infty}.$$

Hence,

$$\lim_{k\to\infty} \sum_{j=k}^{\infty} g_D(z, w_j) = 0, \quad z \in D. \qquad \blacksquare$$

THEOREM 20: *Let D, G be domains in $\mathbb{C}$ and let $h: D \to G$ be a holomorphic covering. Assume that ∂G is polar. Then for any $w_0 \in G$ and any neighborhood V_0 of w_0 such that $h^{-1}(V_0) = \bigcup_{j=1}^{\infty} V_j$, where V_j are disjoint open sets, we have*

$$\omega(z, \bigcup_{j=k}^{\infty} V_j, D) = -1, \quad z \in D, \ k \geq 1. \tag{3}$$

Therefore, h is not an A-covering.

Proof: By the Liouville theorem, any bounded above subharmonic function on G must be constant. Hence, $\omega(z, F, G) = -1$ for any non-empty set $F \subset G$.

If ∂D is polar then (3) is immediate. So, we may assume that ∂D is non-polar. It is well-known that in this case, for any relatively compact set $F \Subset D$ there exists a polar set $P \subset \partial D$ such that

$$\lim_{z \to \partial D \setminus P} \omega(z, F, D) = 0.$$

We know that (see, e.g., [Lár-Sig])

$$\omega(z, \bigcup_{j=1}^{\infty} V_j, D) = \omega(h(z), \mathbb{D}(a, r), G) = -1, \quad z \in D.$$

We set $u(z) = \omega(z, \bigcup_{j=1}^{k-1} V_j, D)$ and $v(z) = \omega(z, \bigcup_{j=k}^{\infty} V_j, D)$, $z \in D$. Note that

$$-1 = \omega(z, \bigcup_{j=1}^{\infty} V_j, D) \geq u(z) + v(z)$$

and that $\bigcup_{j=1}^{k-1} V_j \Subset D$. Therefore, there exists a polar set $P \subset \partial D$ such that

$$\lim_{z \to \partial D \setminus P} \omega(z, \bigcup_{j=1}^{k-1} V_j, D) = 0.$$

Hence,

$$\limsup_{z \to \partial D \setminus P} v(z) \leq -1$$

and, therefore, $v \leq -1$. On the other hand $v \geq -1$. So, $v \equiv -1$. ∎

As a corollary of Theorems 13 and 19 we have the following.

COROLLARY 21: *Let $D_1, \ldots, D_n$ be domains in $\mathbb{C}$ such that $\partial D_1, \ldots, \partial D_n$ are non-polar. Put $D = D_1 \times \ldots \times D_n$. Then for any pluripolar set $P \subset D$ we have*

$$P_D^* = P_D^-.$$

Proof of Corollary 21: Take universal coverings $\pi_i\colon \mathbb{D} \to D_i$, $i = 1, \ldots, n$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. We put $\pi = (\pi_1, \ldots, \pi_n)$. Note that $\pi\colon \mathbb{D}^n \to D$ is an A-covering. We put $Q := \pi^{-1}(P)$. According to Theorems 1, 6 and 13 we have (recall that $\mathbb{D}^n$ is hyperconvex)

$$P_D^* = \pi(Q)_D^* \overset{\text{Thm. 6}}{\supset} \pi(Q_{\mathbb{D}^n}^*) \overset{\text{Thm. 1}}{=} \pi(Q_{\mathbb{D}^n}^-) \overset{\text{Thm. 13}}{=} \pi(Q)_D^- = P_D^- \supset P_D^*.$$

Hence, $P_D^* = P_D^-$. ∎

Proof of Theorem 5: By Corollary 21, it suffices to prove $h(P)_G^- = h(P_D^-)$.

Take universal coverings $\pi_i\colon \mathbb{D} \to D_i$, $i = 1, \ldots, n$. We put $\pi = (\pi_1, \ldots, \pi_n)$ and $\rho = (h_1 \circ \pi_1, \ldots, h_n \circ \pi_n)$. Note that $\rho\colon \mathbb{D}^n \to G$ is an A-covering. We put $Q := \pi^{-1}(P)$. According to Theorem 13 we have

$$\rho(Q_{\mathbb{D}^n}^-) = \rho(Q)_G^- = h(P)_G^-.$$

It remains to note that π is also an A-covering. Hence,

$$\rho(Q_{\mathbb{D}^n}^-) = h \circ \pi(Q_{\mathbb{D}^n}^-) = h(\pi(Q)_D^-) = h(P_D^-). \quad ∎$$

5. Extension through closed pluripolar sets

The main result of this part is the following technical, nevertheless very useful result, special cases of which appeared in [Lev-Pol], [Wie1].

THEOREM 22: *Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let P be a locally closed pluripolar subset of D. Suppose that S is a closed subset of D such that $P \cap S = \emptyset$ and that $D_1 \Subset D_2 \Subset \cdots \Subset D$, $\bigcup_\nu D_\nu = D$ is a sequence of domains. Suppose also that for any $z_0 \in P$ there exists a neighborhood U_0 of z_0 such that*

$$\omega(z, P \cap U_0 \cap D_\nu, D_\nu) \to 0 \quad \text{when } D_\nu \setminus S \ni z \to S. \tag{4}$$

Then $P_D^* \cap S = \emptyset$.

Before we proceed with the proof of Theorem 22, recall the following result.

THEOREM 23 (see [Lev-Pol]): *Let D, G be domains in $\mathbb{C}^n$ and let $D \Subset G$. Assume that $P \subset D$ is a compact set, $z_0 \in D$, V is a neighborhood of z_0 in G, and $V \cap P = \emptyset$. Let $K = \overline{\partial V \cap D}$. If $\omega(z_0, P, D) = -\epsilon$, $\epsilon > 0$, then there is a point $w_0 \in K$ such that $\omega(w_0, P, G) \leq -\epsilon$.*

Proof of Theorem 22: Fix $z_0 \in P$. Then there exists ν_0 such that $z_0 \in D_{\nu_0}$. Without loss of generality, we may assume that $\nu_0 = 1$. There exists $r > 0$ such that $U_0 := \{z \in \mathbb{C}^n : |z - z_0| < 2r\} \Subset D_1$ and condition (4) is fulfilled. We put $P_0 := \{z \in P : |z - z_0| \leq r\}$. Note that P_0 is a compact set and $P_0 \subset P \cap U_0$.

Fix $\nu \geq 1$ and $w_0 \in S$. Suppose that $\omega(w_0, P_0, D_\nu) = -\epsilon$, $\epsilon > 0$. There exists $\delta > 0$ such that $\omega(w, P \cap U_0 \cap D_{\nu+1}, D_{\nu+1}) \geq -\epsilon/2$ for $w \in \overline{V}$, where $V = \{w \in \mathbb{C}^n : \operatorname{dist}(w, S \cap D_\nu) < \delta\}$. Then by Theorem 23 there exists $w_1 \in \overline{\partial V \cap D_\nu}$ such that $\omega(w_1, P_0, D_{\nu+1}) \leq -\epsilon$. Contradiction. Hence, $\omega(w_0, P_0, D_\nu) = 0$. So, $\omega(w_0, P \cap U_0, D_\nu) = 0$. From this we have $w_0 \notin (P \cap U_0)^-_{D_\nu}$ and, therefore, $w_0 \notin \bigcup_\nu (P \cap D_\nu)^-_{D_\nu} = P^*_D$. ∎

6. Some examples and applications

At first we give an example in which we show that there exists a holomorphic covering and a pluripolar set such that (2) does not hold.

Example 24: Let K be a closed polar set in $\mathbb{C}$ such that $\#K \geq 2$. Put $G_1 := \mathbb{C} \setminus K$. Suppose that $h_1: \mathbb{D} \to G_1$ is a universal covering. Put

$$P := \{(z^2, z) : z \in \mathbb{D}\} \subset \mathbb{D} \times \mathbb{D}$$

and $h_2 = \mathrm{id}: \mathbb{D} \to G_2$, where $G_2 = \mathbb{D}$. Then $P^-_{\mathbb{D}^2} = P$ and, therefore, $h(P^-_{\mathbb{D}^2}) = h(P)$. But, $h(P)^-_G = G$.

Now we consider a special holomorphic covering.

THEOREM 25: *Let $\pi: \mathbb{C}^2 \ni (z, w) \mapsto (z, e^w) \in \mathbb{C}^2$. Assume that $P \subset \mathbb{C}^2$ is a pluripolar set. Then*

$$[\pi(P)]^*_{\mathbb{C}\times\mathbb{C}_*} = \left([\pi(P)]^*_{\mathbb{C}^2}\right) \cap (\mathbb{C} \times \mathbb{C}_*) = \pi(P^*_{\mathbb{C}^2}),$$

where $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$.

Moreover, if P is a locally closed and for any $w_0 \in \pi(P)$ there exists a neighborhood $U_0 \subset \mathbb{C}^2$ of w_0 such that $\pi^{-1}(\pi(P) \cap U_0) \Subset \mathbb{C}^2$, then $[\pi(P)]^*_{\mathbb{C}^2} = \pi(P)$.

Theorem 25 is a generalization of results from [Lev-Pol] and [Wiel].

Proof of Theorem 25: Put $D_\nu := \{(\lambda, \zeta) \in \mathbb{C}^2 : |\lambda| < \nu, \operatorname{Re}\zeta < \log\nu\}$ and $G_\nu := \{(\lambda, \zeta) \in \mathbb{C}^2 : |\lambda| < 1, |\zeta| < 1\}$, $\nu > 0$. Note that $\pi\colon D_\nu \to G_\nu \setminus A$, where $A := \mathbb{C} \times \{0\}$, is an A-covering and D_ν is simply connected. Hence,

$$[\pi(P \cap D_\nu)]^-_{G_\nu \setminus A} = \pi((P \cap D_\nu)^-_{D_\nu}).$$

Therefore,

$$\begin{aligned}
[\pi(P)]^*_{\mathbb{C}\times\mathbb{C}_*} &\subset \big([\pi(P)]^*_{\mathbb{C}^2}\big) \cap \big(\mathbb{C} \times \mathbb{C}_*\big) \\
&= \Big(\bigcup_{\nu=1}^{\infty} [\pi(P) \cap G_\nu]^-_{G_\nu}\Big) \cap (\mathbb{C} \times \mathbb{C}_*) = \Big(\bigcup_{\nu=1}^{\infty} [\pi(P \cap D_\nu)]^-_{G_\nu}\Big) \bigcap (\mathbb{C} \times \mathbb{C}_*) \\
&= \bigcup_{\nu=1}^{\infty} [\pi(P \cap D_\nu)]^-_{G_\nu \setminus A} = \bigcup_{\nu=1}^{\infty} \pi((P \cap D_\nu)^-_{D_\nu}) = \bigcup_{\nu=1}^{\infty} \pi((P \cap D_\nu)^*_{D_\nu}) \\
&= \pi\Big(\bigcup_{\nu=1}^{\infty} [P \cap D_\nu]^*_{D_\nu}\Big) \subset \pi(P^*_{\mathbb{C}^2}) \subset [\pi(P)]^*_{\mathbb{C}\times\mathbb{C}_*}.
\end{aligned}$$

For the second part, we have to show that

$$\pi(P)^*_{\mathbb{C}^2} \cap A = \emptyset. \tag{5}$$

There exist compact sets K_j such that $P = \bigcup_{j=1}^{\infty} K_j$. Note that

$$\pi(P)^*_{\mathbb{C}^2} = \bigcup_{j=1}^{\infty} (\pi(K_j))^*_{\mathbb{C}^2}.$$

So, (5) is equivalent to $\pi(K_j)^*_{\mathbb{C}^2} \cap A = \emptyset$, $j \geq 1$.

Fix j and put $K = K_j$. By Theorem 22 it suffices to show

$$\omega(w, K \cap U_0 \cap D_\nu, D_\nu) \to 0 \quad \text{when } w \to A, \nu \geq 1,$$

where U_0 is a fixed neighborhood of $w_0 \in \pi(K)$. We know that there exists a neighborhood U_0 of w_0 such that

$$\omega(z, \pi^{-1}(\pi(K) \cap U_0), D_\nu) = \omega(\pi(z), K \cap U_0, D_\nu).$$

Suppose that $z = (\lambda, \zeta)$. Note that $\pi(z) \to 0$ if and only if $\operatorname{Re}\zeta \to -\infty$. It suffices to note that

$$\omega(z, \pi^{-1}(\pi(K) \cap U_0), D_\nu) \geq \omega(\rho(z), \rho(\pi^{-1}(\pi(K) \cap U_0)), H_\nu) \to 0 \qquad \text{when } \operatorname{Re}\rho(z) \to -\infty,$$

where $\rho\colon \mathbb{C}^2 \ni (\lambda, \zeta) \to \zeta \in \mathbb{C}$ is a projection. ∎

COROLLARY 26: (a) *Let ϕ be a holomorphic function on $\mathbb{C}$. Put*

$$Q := \{(\phi(\lambda), e^{\lambda}) : \lambda \in \mathbb{C}\}.$$

If $(\phi, \exp)\colon \mathbb{C} \to \mathbb{C}^2$ is injective, then $Q^*_{\mathbb{C}^2} = Q$.

(b) *Let ψ be a meromorphic function on $\mathbb{C}$. Put*

$$R := \{(\lambda, e^{\psi(\lambda)})\colon \lambda \in \mathbb{C} \setminus \{\psi^{-1}(\infty)\}\}.$$

Then $R^*_{\mathbb{C}^2} = R$ and, therefore, R is a complete pluripolar set in $\mathbb{C}^2$.

Proof of Corollary 26: Easily follows from Theorems 25 and 2. ∎

Let S be a subset in $\mathbb{C}^n$. We say that S is **pluri-thin at** $z_0 \in \mathbb{C}^n$ if $z_0 \notin \overline{S} \setminus S$ or there exists a plurisubharmonic function $u \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\limsup_{z \to z_0, z \in E \setminus \{z_0\}} u(z) < u(z_0).$$

It is well-known that a real-analytic curve is not pluri-thin at each of its points (see, e.g., [Sad]).

The following simple remark shows a relation between pluri-thin points and pluripolar hull.

Remark 27: Let P be a pluripolar set in $\mathbb{C}^n$. Then P is pluri-thin at any point of $\overline{P} \setminus P^*_{\mathbb{C}^n}$.

Proof of Remark 27: Assume that $z_0 \in \overline{P} \setminus P^*_{\mathbb{C}^n}$. Then there exists a psh function u on $\mathbb{C}^n$ such that $u|_P \equiv -\infty$ and $u(z_0) > -\infty$.

A. Sadullaev (see [Sad]) asked whether the following sets are pluri-thin at the origin:

$$P_1 := \{(t^{\alpha}, t) : 0 \le t \le 1\} \subset \mathbb{C}^2, \tag{6}$$

where $\alpha > 0$ is irrational, and

$$P_2 := \{(t, e^{1/t}) : -1 \le t < 0\}. \tag{7}$$

Note that the existence of a plurisubharmonic function $u \in \mathrm{PSH}(\mathbb{C}^2)$ such that $P_1 \subset \{z \in \mathbb{C}^2 : u(z) = -\infty\}$ (respectively, $P_2 \subset \{z \in \mathbb{C}^2 : u(z) = -\infty\}$) such that $u(0) > -\infty$ will solve positively A. Sadullaev's question.

Let us consider the following sets:

$$F_1 := \{(e^{\alpha\lambda}, \lambda) : \lambda \in \mathbb{C}\} \subset \mathbb{C}^2,$$

where $\alpha > 0$ is irrational, and

$$F_2 := \{(\lambda, 1/\lambda) : \lambda \in \mathbb{C}_*\} \subset \mathbb{C}^2.$$

COROLLARY 28: (a) $[\pi(F_1)]^*_{\mathbb{C}^2} = \pi(F_1)$. (b) *The set* $\pi(F_2)$ *is complete pluripolar in* $\mathbb{C}^2$.

So, Corollary 28 gives a positive answer to both questions of A. Sadullaev. Part (a) of Corollary 28 was proved by N. Levenberg and E. A. Poletsky [Lev-Pol] and part (b) by J. Wiegerinck [Wie1, Wie2].

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